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Maximum genus and chromatic number of graphs ☆

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Abstract

Let T be a spanning tree of a connected graph G. Denote by $\xi(G,T)$ the number of components in $G\backslash E(T)$ with odd number of edges. The value $\min_T \xi(G,T)$ is known as the *Betti deficiency* of G, denoted by $\xi(G)$, where the minimum is taken over all spanning trees T of G. It is known (N.H. Xuong, J. Combin. Theory 26 (1979) 217–225) that the maximum genus of a graph is mainly determined by its Betti deficiency $\xi(G)$. Let G be a k-edge-connected graph ($k \leqslant 3$) whose complementary graph has the chromatic number m. In this paper we prove that the Betti deficiency $\xi(G)$ is bounded by a function $f_k(m)$ on m, and the bound is the best possible. Thus by Xuong's maximum genus theorem we obtain some new results on the lower bounds of the maximum genus of graphs.

Keywords: Maximum genus; Betti deficiency; Chromatic number

1. Introduction

All graphs in this paper are finite and furthermore simple. Terminology and notation without explicit explanation are those of [2]. The *chromatic number*, denoted by $\chi(G)$, of a graph G is the smallest number of colors for $V(G)$ so that adjacent vertices are colored differently. For a graph G, its *complementary graph*, denoted by G^c, is the graph with vertex set $V(G)$ such that two vertices are adjacent in G^c if and only if these vertices are not adjacent in G. For any edge subset X of a graph G, $G\backslash X$ is the graph obtained from G by removing all edges in X.

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By a *surface*, we will mean a compact and connected 2-manifold with no boundaries. It is well known from elementary topology that a surface can be classified as an orientable one or a nonorientable one. An orientable surface can be viewed as a sphere with h handles attached to it, while a nonorientable surface as a sphere with k crosscaps attached to it. The number h or k is called the *genus* of the surface according to the orientability. A graph is said to be *embedded* in a surface S if it is "drawn" in S so that edges intersect only at their common vertices. Recall that the *maximum genus*, denoted by $\gamma_M(G)$, of a connected graph G is defined as the maximum integer k such that there exists a cellular embedding of G in an orientable surface S of genus k. That an embedding of G in a surface S is called *cellular* means that the complement $S\backslash G$ is the union of disjoint open 2-cells. Note that a disconnected graph does not admit a cellular embedding in any surface. Since any cellular embedding of a connected graph G must have at least one face, Euler's formula gives that $\gamma_M(G) \leqslant \lfloor \beta(G)/2 \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as the *cyclic number* of the connected graph G (for any real number x, $\lfloor x \rfloor$ denotes the maximum integer no greater than x). A connected graph G is called *upper embeddable* if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$.

The maximum genus, a topological invariant of graphs, involving the cellular embedding of a graph on an orientable surface of as high as possible genus, was introduced by Nordhaus et al. [14]. A previous result in [18] states that each 4-edge-connected graph is upper embeddable. However, the somewhat weaker hypothesis of being 3-edge-connected does not guarantee the upper embeddablity of graphs, for indeed there exist many such graphs that are not upper embeddable (see [7] for example). On the one hand, based on other restricting conditions one wishes to find some distinct classes of upper embeddable graphs (see papers [6,10–13,16]); on the other hand, one wishes to find good lower bounds on the maximum genus of graphs. In the study of the lower bounds on the maximum genus of graphs, if we emphasize the edge-connectivity condition, by the result in [18] we need only consider such graphs with edge-connectivity $\leqslant 3$. Based on this aim, papers [4,8] give some results on this aspect. Using other invariants of graphs, such as diameter, girth, independent number, papers [1,15,19–22] have provided some lower bounds on the maximum genus of graphs.

In this paper we continue to study lower bounds on the maximum genus of graphs. But here we are doing so by combining not only the edge-connectivity but also the chromatic number of graphs, an invariant which has intensely been investigated in the coloring theory of graphs. It is known from [18] that the maximum genus of a graph G depends essentially on the *Betti deficiency* $\xi(G)$ (whose definition is explained in the abstract), see Lemma 1 below. Let G be a connected graph with the edge-connectivity $k \leqslant 3$ and with $\chi(G^c) = m$. In this paper we prove that the Betti deficiency $\xi(G)$ is bounded by a function $f_k(m)$ on m. We also show by examples that the upper bound $f_k(m)$ is the best possible. Thus by Xuong's maximum genus theorem, the upper bound $f_k(m)$ on $\xi(G)$ can be immediately translated into a lower bound on the maximum genus of G. As applications of our results we give some new lower bounds on the maximum genus of graphs. A simple outline of this paper is as follows. In Section 2 we state some basic results on the maximum genus of graphs. The main results are given in Section 3, while Section 4 contains some applications.

2. Some basic results

The following first result, due to Xuong [18], closely relates the maximum genus to the Betti deficiency of a graph.

Lemma 1. *Let G be a connected graph. Then* (1) $\gamma_M(G) = [\beta(G) - \xi(G)]/2$; (2) *$G$ is upper embeddable if and only if $\xi(G) \leqslant 1$.*

From Lemma 1 above, the maximum genus of a graph G is mainly determined by the Betti deficiency $\xi(G)$, since the cyclic number $\beta(G)$ can be easily computed.

Again, for a graph G and $A \subseteq E(G)$, denote by $c(G\backslash A)$ the number of the components of $G\backslash A$ and by $b(G\backslash A)$ the number of the components of $G\backslash A$ with odd cyclic number. The following result was proved by Nebeský [9].

Lemma 2. *Let G be a connected graph. Then*

$$\xi(G) = \max_{A \subseteq E(G)} \{c(G\backslash A) + b(G\backslash A) - |A| - 1\}.$$

Let $F_1, F_2, \ldots, F_l$ be l ($l \geqslant 2$) distinct subgraphs of G. Then denote by $E_G(F_1, F_2, \ldots, F_l)$ the set of those edges of G whose two end vertices are respectively in two pairwise subgraphs F_i and F_j for $1 \leqslant i, j \leqslant l$ and $i \neq j$.

Lemma 3. *Let G be a connected graph. If $\xi(G) \geqslant 2$, namely G is not upper embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties hold*:

(i) $c(G\backslash A) = b(G\backslash A) \geqslant 2$;
(ii) *any component F of $G\backslash A$ is a vertex-induced subgraph of G*;
(iii) $|E_G(F_1, F_2, \ldots, F_l)| \leqslant 2l - 3$ *for any $l \geqslant 2$ distinct components $F_1, F_2, \ldots, F_l$ of $G\backslash A$*;
(iv) $\xi(G) = 2c(G\backslash A) - |A| - 1$.

Proof. By Lemma 2 there exists a subset $A \subseteq E(G)$ so that

$$\xi(G) = c(G\backslash A) + b(G\backslash A) - |A| - 1.$$

Therefore, we may assume that the chosen subset A in the above equation has the minimum number of edges. Then the detailed proofs of properties (i)–(iii) are given in [21], while (iv) follows from (i) and the choice of A. □

By Lemma 3, we further have the following result:

Lemma 4 (main lemma). *Under the conditions and conclusions of Lemma 3 we have*

(1) $\xi(G) \leqslant c(G\backslash A)$;

(2) $\xi(G) \leqslant c(G\backslash A) - 1$ *if* G *is* 2-*edge-connected*;
(3) $\xi(G) \leqslant \lfloor \frac{c(G\backslash A)}{2} \rfloor - 1$ *if* G *is* 3-*edge-connected.*

Proof. According to Lemma 3 we construct a graph $G' = G/(G\backslash A)$ as follows. The vertices of G' are the components of $G\backslash A$. For each edge in A joining a pair of components in $G\backslash A$, we make an edge in G' joining the corresponding vertices. It is easy to see that G' is connected because of the connectivity of G. Furthermore, G' is simple by property (iii) of Lemma 3 (for $l = 2$). By the connectivity of G', we have that $|A| = |E(G')| \geqslant |V(G')| - 1 = c(G\backslash A) - 1$, and thus Part (1) immediately follows from property (iv) of Lemma 3. Again, for any component F of $G\backslash A$, let $e(F, G)$ be the number of edges whose one end vertex is in F while the other one is not in F. From the definition of G', we see that the degree of each vertex of G', corresponding to a component F of $G\backslash A$, equals $|e(F, G)|$. If G is 2-edge-connected, then $e(F, G) \geqslant 2$ for any component F of $G\backslash A$, and so it follows that each vertex in G' has the degree at least 2. Therefore, the edge-degree relation of graphs ensures that $|A| = |E(G')| \geqslant |V(G')| = c(G\backslash A)$. By property (iv) of Lemma 3 we get Part (2). Noting that $\xi(G)$ is an integer, similarly we easily obtain Part (3). □

3. The main results

Since we are studying lower bounds on the maximum genus of graphs, just as mentioned above we consider only such graphs with edge-connectivity $\leqslant 3$. By Lemma 1, a better upper bound on the Betti deficiency $\xi(G)$ of a graph G determines a better lower bound on $\gamma_M(G)$ of G. The following theorem gives some upper bounds on $\xi(G)$ of a graph G in connection with the chromatic number $\chi(G^c)$.

Theorem 1. *Let* G *be a connected graph with the edge-connectivity* $k \leqslant 3$. *If* $\chi(G^c) = m$, *then certain upper bounds on the Betti deficiency* $\xi(G)$ *are given in the following table.*

$k = 1$	$k = 2$	$k = 3$
m	$\max\{m - 1, 1\}$	$\max\{\lfloor \frac{m}{2} \rfloor - 1, 1\}$

Proof. Note first that the hypothesis $\chi(G^c) = m$ is equivalent to assuming that m is the smallest number for which there exists a partition of $V(G)$ into sets $V_1, V_2, \ldots, V_m$ such that the vertex-induced subgraph $G[V_i]$ of G is a complete graph for each i, $1 \leqslant i \leqslant m$. We see that the conclusions of the theorem are trivial if $\xi(G) \leqslant 1$ because $m \geqslant 1$, and thus we now assume that $\xi(G) \geqslant 2$. Since $\xi(G) \geqslant 2$, it follows from Lemma 3 that there exists a subset $A \subseteq E(G)$ such that all the properties of Lemma 3 are satisfied. Thus by Lemma 4, in order to obtain the proof of the theorem it suffices to prove that $c(G\backslash A) \leqslant m$. We first have the following claims:

Claim 1. *For any V_i $(1 \leqslant i \leqslant m)$ and any component F of $G\backslash A$, if $V_i \nsubseteq V(F)$ then either $|V_i \cap V(F)| = 0$ or $|V_i \cap V(F)| = 1$.*

Proof. Assume to the contrary that it is not the case. Then there exists a component H of $G\backslash A$ where $H \neq F$ and a vertex $x \in V_i$ such that $x \in V_i \bigcap V(H)$, and furthermore there exist two distinct vertices $y, z \in V_i \cap V(F)$. Since $G[V_i]$ is a complete graph, obviously $xy, xz \in E(G)$, and thus $xy, xz \in A$. This thus implies that $|E_G(F,H)| \geqslant 2$, contradicting the property (iii) of Lemma 3 for $l = 2$. Hence the claim is true. □

Claim 2. *For any V_i $(1 \leqslant i \leqslant m)$ and any component F of $G\backslash A$, if $V_i \nsubseteq V(F)$ then either $|V_i| = 2$ or $|V_i| = 3$.*

Proof. Let $V_i = \{x_1, x_2, \ldots, x_l\}$. See first that $l \geqslant 2$ for otherwise V_i must belong to the vertex set of some component of $G\backslash A$, contradicting the assumption of the claim. By Claim 1, there exist l distinct components $F_1, F_2, \ldots, F_l$ of $G\backslash A$ such that $V_i \cap V(F_j) = \{x_j\}$, $j = 1, 2, \ldots, l$. Note that each edge of $G[V_i]$ belongs to $E_G(F_1, F_2, \ldots, F_l)$. Since $G[V_i]$ is a complete graph, it follows that

$$|E_G(F_1, F_2, \ldots, F_l)| \geqslant |E(G[V_i])| = \frac{1}{2}\, l(l-1).$$

By the property (iii) of Lemma 3 we get that $2l - 3 \geqslant \frac{1}{2} l(l-1)$, namely $2 \leqslant l \leqslant 3$, as desired. □

Claim 3. $|V(F)| \geqslant 3$ *for each component F of $G\backslash A$.*

Proof. It is straightforward because G and thus F are simple and because $\beta(F) = 1$ (mod 2) by property (i) of Lemma 3. □

Proof of Theorem 1 (Conclusion). Now denote by $\mathscr{F}$ the set of all components of $G\backslash A$ and let $\mathscr{V} = \{V_1, V_2, \ldots, V_m\}$. Keep in mind that in order to prove the lemma we only need to show that $|\mathscr{F}| = c(G\backslash A) \leqslant m = |\mathscr{V}|$. Let

$$\mathscr{F}_0 = \{F\colon F \in \mathscr{F} \text{ and there exists some } V_i \subseteq V(F) \text{ for } 1 \leqslant i \leqslant m\},$$

and

$$\mathscr{V}_0 = \{V_i\colon V_i \subseteq V(F) \text{ for some component } F \text{ of } G\backslash A \text{ and } 1 \leqslant i \leqslant m\}.$$

Clearly $|\mathscr{F}_0| \leqslant |\mathscr{V}_0|$. Let $\mathscr{F}_1 = F\backslash\mathscr{F}_0$ and $\mathscr{V}_1 = \mathscr{V}\backslash\mathscr{V}_0$. We then only show that $|\mathscr{F}_1| \leqslant |\mathscr{V}_1|$. Now define a bipartite graph G^* (possibly disconnected) as follows: the vertices of G^* are viewed as all the elements in $\mathscr{F}$ or $\mathscr{V}_1$, and the two parts of vertices of G^* are $\mathscr{F}$ and $\mathscr{V}_1$; put an edge joining a vertex F in $\mathscr{F}$ and a vertex V_i in $\mathscr{V}_1$ if and only if $|V_i \cap V(F)| = 1$ by Claim 1.

Denote by E_1 (resp. E_2) all such edges of G^* incident to the vertices in $\mathscr{V}_1$ (resp. $\mathscr{F}_1$). Note that the graph G^* has the following simple properties:

Property 1. Each vertex V_i in $\mathscr{V}_1$ has the degree 2 or 3 according to as $|V_i| = 2$ or $|V_i| = 3$ by the definition of $\mathscr{V}_1$ and by Claims 1 and 2, and so $|E_1| \leqslant 3|\mathscr{V}_1|$.

Property 2. Each vertex F in $\mathscr{F}_1$ has degree $|V_i| \in \{2,3\}$ by the definition of $\mathscr{F}_1$ and by Claims 1 and 3, and thus $|E_2| \geqslant 3|\mathscr{F}_1|$.

Since G^* is a bipartite graph, we have $E_2 \subseteq E_1$, and thus

$$3|\mathscr{F}_1| \leqslant |E_2| \leqslant |E_1| \leqslant 3|\mathscr{V}_1|,$$

showing that $|\mathscr{F}_1| \leqslant |\mathscr{V}_1|$. Thereby we eventually obtain that $|\mathscr{F}| \leqslant |\mathscr{V}|$ and then complete the proof of the theorem by the above analysis. □

Now we shall discuss the sharpness of these upper bounds on $\xi(G)$ given in Theorem 1.

Theorem 2. *The upper bounds given in Theorem* 1 *can be achieved by infinite many graphs G with arbitrarily large value $m = \chi(G^c)$.*

Proof. Let H be an arbitrary connected graph with the maximum degree at most 3, and let G be a complete graph with odd $\beta(G)$ and with a large number of vertices (for our technical reason, assume that $|V(G)| \geqslant 4$). From H and G we define a new graph, denoted by $H \otimes G$. The graph $H \otimes G$ is obtained as follows: first replace each vertex $v \in V(H)$ by the graph G, and then put all of the edges of H previously incident upon v to be incident upon distinct vertices in G (this can be guaranteed because $|V(G)| \geqslant 4$ and $d_H(v) \leqslant 3$ for each vertex v of H). By its definition the graph $H \otimes G$ is not unique. It is easy to see that $H \otimes G$ is a connected simple graph, and furthermore that all such edges of $H \otimes G$ not lying in a copy of G correspond to the initial edges of H. We now verify that the graph $H \otimes G$ has the following properties:

Property 1. *$H \otimes G$ is k-edge-connected, if H is k-edge-connected for $k = 1,2,3$.*

Proof. It is clear from the definition. □

Property 2. $\chi((H \otimes G)^c) = |V(H)|$.

Proof. We first easily show that the complementary graph $(H \otimes G)^c$ is $|V(H)|$-colorable. Suppose $\chi((H \times H)^c) < |V(H)|$. Then there must exist some vertex subset V_i of the graph $(H \otimes G)^c$ corresponding to some color i so that $|V_i| \geqslant |V(G)| + 1$, because the number of vertices of $H \otimes G$ is $|V(H)| \cdot |V(G)|$. We notice that in the graph $H \otimes G$, the vertex-induced subgraph $(H \otimes G)[V_i]$ is a complete graph. However, it is observed from the construction of $H \otimes G$ that the number of vertices of the maximum complete subgraph in $H \otimes G$ is just $|V(G)|$. This is a contradiction. □

Property 3. $\xi(H \otimes G) \geqslant 2|V(H)| - |E(H)| - 1$.

Proof. Denote by A the edge subset of $H \otimes G$ corresponding to all initial edges in H. Thus $|A| = |E(H)|$. We observe that $(H \otimes G)\backslash A$ has exactly $|V(H)|$ components, each being a copy of G. Note that $\beta(G)$ is odd. This implies that $c((H \otimes G)\backslash A) = b((H \otimes G)\backslash A) = |V(H)|$. Therefore the conclusion is immediately obtained from Lemma 2. □

For arbitrarily large value m, in the following we choose an appropriate graph H such that $\chi((H \otimes G)^c) = m$ and the Betti deficiency $\xi(H \otimes G)$ arrives at the given upper bounds in Theorem 1. Apparently there are infinitely many such graphs $H \otimes G$. We distinguish the following three cases according to the edge-connectivity $k = 1, 2, 3$, respectively.

(1) Let H be a tree with $m \geqslant 2$ vertices. By Property 1, $H \otimes G$ is 1-edge-connected, for H is 1-edge-connected. By Property 2, $\chi((H \otimes G)^c) = m$. By Property 3 and Theorem 1 we have $\xi(H \otimes G) = 2|V(H)| - |E(H)| - 1 = m$. Therefore $H \otimes G$ is the desired graph for the case of edge-connectivity $k = 1$ in Theorem 1.

(2) Let H be a circuit with $m \geqslant 3$ vertices. Similarly, we easily check that $H \otimes G$ is the desired graph for the case of edge-connectivity $k = 2$ in Theorem 1.

(3) Let H be a 3-edge-connected 3-regular simple graph, and let $|V(H)| = m$ if $m \geqslant 4$ is even, and $|V(H)| = m - 1$ if $m \geqslant 4$ is odd. Thus the upper bound on ξ in Theorem 1 for the case of edge-connectivity $k = 3$ is $m/2 - 1$ if m is even, and $(m-1)/2 - 1$ if m is odd. By Property 1, $H \otimes G$ is 3-edge-connected. By Property 2, $\chi((H \otimes G)) = m$ if m is even, and $\chi((H \otimes G)) = m - 1$ if m is odd. By Property 3, $\xi(H \otimes G) \geqslant 2|V(H)| - |E(H)| - 1 = \frac{1}{2}|V(H)| - 1$. At last we get from Theorem 1 that $\xi(H \otimes G) = m/2 - 1$ if m is even, and $\xi(H \otimes G) = (m-1)/2 - 1$ if m is odd. This shows that $H \otimes G$ is as desired.

From the above analysis the proof of the theorem is finished. □

According to Theorems 1 and 2, we have the following theorem.

Theorem 3. *Let G be a connected graph with the edge-connectivity $k\,(\leqslant 3)$ and $\chi(G^c) = m$. Then lower bounds on the maximum genus $\gamma_M(G)$ are given in the following table, and the lower bounds are the best possible.*

$k = 1$	$k = 2$	$k = 3$
$\dfrac{\beta(G) - m}{2}$	$\dfrac{\beta(G) - \max\{m-1,\, 1\}}{2}$	$\dfrac{\beta(G) - \max\{\lfloor \frac{m}{2} \rfloor - 1,\, 1\}}{2}$

Proof. By Lemma 1, the claim is an immediate consequence of Theorems 1 and 2. □

Note: The lower bounds on the maximum genus of G given in Theorem 3 are also true when k is the vertex-connectivity, for vertex-connectivity implies edge-connectivity of graphs.

We see that our results in Theorem 3 are different from those in [5], where it is proved that $\beta(G)/4$ (or $\beta(G)/3$, respectively) is a tight lower bound on the maximum genus of a simple graph G with the minimal degree $\geqslant 3$ and with the edge-connectivity 1 (or, the edge-connectivity 2 or 3, respectively). It follows from Theorem 3 that the results in [5] can be greatly improved, if the complementary graph has small chromatic number, for example if the complementary graph is a planar graph. We will give these results in detail in the next section.

4. Some consequences

With the aid of some known results on the chromatic number of graphs, in this section we give some lower bounds on the maximum genus of a graph in terms of other parameters. Based on the same reason that each 4-edge-connected graph is upper embeddable, we here only consider graphs with edge-connectivity $k \leqslant 3$.

The following first corollary establishes lower bounds on the maximum genus of a graph in terms of its degree sequences.

Corollary 1. *Let G be a k-edge-connected graph of p vertices having the degree sequence $(d_1, d_2, \ldots, d_p)$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$. Let $d = \max_{1 \leqslant i \leqslant p}\{\min\{p - d_{p+1-i}, i\}\}$. Then*

(1) *if* $k = 1$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - d)$;
(2) *if* $k = 2$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{d - 1, 1\})$;
(3) *if* $k = 3$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{\frac{1}{2}(d - 1), 1\})$.

Proof. The complement G^c has the nondecreasing degree sequence $(p - 1 - d_p, p - 1 - d_{p-1}, \ldots, p - 1 - d_1)$. Thus by a known result (see Theorem 10.5 of Chapter 10 of [3]), we have

$$\chi(G^c) \leqslant \max_{1 \leqslant i \leqslant p}\{\min\{1 + (p - 1 - d_{p+1-i}), i\}\} = d.$$

Thus the conclusion is directly from Theorem 3. □

Let $\Delta(K)$ and $\delta(K)$ denote the maximum and minimum degree in a graph K, respectively. The following result gives lower bounds on the maximum genus in terms of the minimum degree of a graph.

Corollary 2. *Let G be a k-edge-connected graph of p vertices. Then*

(1) *if* $k = 1$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - (p - 1 - \delta(G)))$;
(2) *if* $k = 2$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{(p - 2 - \delta(G)), 1\})$;
(3) *if* $k = 3$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{\frac{1}{2}(p - 3 - \delta(G)), 1\})$.

Proof. We only note the following inequality (see Corollary 10.2 of Chapter 10 of [3]),

$$\chi(G^c) \leqslant \Delta(G^c) + 1 \quad \text{for the graph } G.$$

and the obvious equality $\Delta(G^c) = p - 1 - \delta(G)$. □

The *independence number*, denoted by $\alpha(G)$, of a graph G is the number of vertices in a maximum independent set of G. A *clique* of a graph G is a vertex subset S of G

such that the vertex-induced subgraph $G[S]$ is complete. The *clique number*, denoted by $\omega(G)$, of the graph G is the number of vertices in a maximum clique. Obviously, S is a clique of a graph G if and only if S is an independent set of G^c, so $\alpha(G) = \omega(G^c)$.

The following corollary gives lower bounds on the maximum genus of a graph G, related to its independence number $\alpha(G)$.

Corollary 3. *Let G be a k-edge-connected graph not containing P_4 (a path of four vertices) as an induced subgraph. Then*

(1) *if* $k = 1$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \alpha(G))$;
(2) *if* $k = 2$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{\alpha(G) - 1, 1\})$;
(3) *if* $k = 3$, $\gamma_M(G) \geqslant \frac{1}{2}(\beta(G) - \max\{\frac{\alpha(G)}{2} - 1, 1\})$.

Proof. Note first the following result stated as Theorem 10.6 of Chapter 10 of [3]:

if a graph G does not contain P_4 as an induced subgraph, then $\chi(G) = \omega(G)$,

meantime note also that if a graph G contains P_4 as induced subgraph if and only if its complement G^c does not. Thus we get that $\chi(G^c) = \omega(G^c) = \alpha(G)$, and the conclusions are clear as well. □

Considering that each planar graph is 5-colorable we have the following result:

Corollary 4. *Let G be a k-edge-connected graph. If G^c is planar, then*

(1) *if* $k = 1$, $\gamma_M(G) \geqslant \frac{\beta(G)-5}{2}$;
(2) *if* $k = 2$, $\gamma_M(G) \geqslant \frac{\beta(G)-4}{2}$;
(3) *if* $k = 3$, $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$, *in other words, G is upper embeddable.*

Proof. Similarly the proof is straightforward. □

We now present a further generalization of Corollary 4 by virtue of the well-known Heawood Map Coloring Theorem. First let us explain some notation and known facts. Recall that the *genus* $\gamma(G)$ (resp. the *nonorientable genus* $\tilde{\gamma}(G)$) of a graph G is defined to be the least integer k such that G can be embedded in an orientable surface S_k of genus k (resp. a nonorientable surface $\tilde{S}_k$ of genus k, formed from a sphere by adding k crosscaps). An embedding of a graph G in an orientable surface S_k of genus k (resp. a nonorientable surface $\tilde{S}_h$ of genus h) is said to be a *minimal orientable* (resp. *nonorientable*) embedding if $\gamma(G) = k$ (resp. $\tilde{\gamma}(G) = h$). The *chromatic number of an orientable surface S_n* (resp. *nonorientable surface $\tilde{S}_n$*) of genus n, denoted by $\chi(S_n)$ (resp. $\chi(\tilde{S}_n)$), is the maximum chromatic number among all graphs that can be embedded in the orientable surface S_n (resp. nonorientable surface $\tilde{S}_n$).

The well-known Heawood Map Coloring Theorem (orientable version) (for example, see [17]) states that

$$\chi(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor \quad \text{for any orientable surface } S_n \text{ of genus } n > 0,$$

while the nonorientable version states that for any nonorientable surface $\tilde{S}_n$ of genus $n \geqslant 0$,

$$\chi(\tilde{S}_n) = \begin{cases} \left\lfloor \frac{7+\sqrt{1+24n}}{2} \right\rfloor & \text{if } n \geqslant 1 \text{ and } n \neq 2; \\ 6 & \text{if } n = 2. \end{cases}$$

Thus for any graph G, if $\gamma(G^c) = n > 0$, it then follows from the definitions that $\chi(G^c) \leqslant \chi(S_n)$. Note that $\chi(S_n) \geqslant 7$ for any orientable surface S_n of genus $n > 0$. Therefore, from Theorem 3 we easily obtain the following result.

Corollary 5. *Let G be a connected graph with edge-connectivity $k \leqslant 3$. If $\gamma(G^c) = n > 0$, then lower bounds on the maximum genus $\gamma_M(G)$ are given in the following table.*

$k = 1$	$k = 2$	$k = 3$
$\frac{1}{2}(\beta(G) - N_n)$	$\frac{1}{2}(\beta(G) - (N_n - 1))$	$\frac{1}{2}(\beta(G) - (\lfloor \frac{N_n}{2} \rfloor - 1))$

where $N_n = \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$.

Likewise, using the nonorientable version of the Map Coloring Theorem we have

Corollary 6. *Let G be a connected graph with edge-connectivity $k \leqslant 3$. If $\tilde{\gamma}(G^c) = n \geqslant 1$, then lower bounds on the maximum genus $\gamma_M(G)$ are given in the following table.*

$k = 1$	$k = 2$	$k = 3$
$\frac{1}{2}(\beta(G) - M_n)$	$\frac{1}{2}(\beta(G) - (M_n - 1))$	$\frac{1}{2}(\beta(G) - (\lfloor \frac{M_n}{2} \rfloor - 1))$

where $M_n = \lfloor \frac{7+\sqrt{1+24n}}{2} \rfloor$ for $n \geqslant 1$ and $n \neq 2$, and $M_2 = 6$.

The above Corollaries 5 and 6 tell us that the respective lower bound on the maximum genus $\gamma_M(G)$ of a graph G increases as the genus of the surface in which the complementary graph G^c can be embedded decreases. For example, let G be a connected graph with edge-connectivity 1, 2 or 3. If the complementary graph G^c can be embedded in the torus, then it follows from Corollary 5 that $\frac{1}{2}(\beta(G)-7)$, $\frac{1}{2}(\beta(G)-6)$, and $\frac{1}{2}(\beta(G) - 2)$, respectively, are lower bounds on $\gamma_M(G)$ for the edge-connectivity k= 1, 2, and 3, respectively. If G^c can be embedded in the projective plane, by Corollary 6, $\frac{1}{2}(\beta(G)-6)$, $\frac{1}{2}(\beta(G)-5)$, and $\frac{1}{2}(\beta(G)-2)$, respectively, are lower bounds on $\gamma_M(G)$ for the edge-connectivity $k = 1, 2$, and 3.We note that these lower bounds are closer to the best upper bound $\lfloor \beta(G)/2 \rfloor$.

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